[T.K] 6.8.1.8 Let $(Y_n)_{n\geq 1}$ be a sequence of random variables such that Y_n is uniformly distributed on the set $\{\frac{j}{2^n} ; j=0,1,2,\dots,2^n-1\}$ $\forall n\geq 1$.

To show : $Y_n \xrightarrow{d} U(0,1)$.

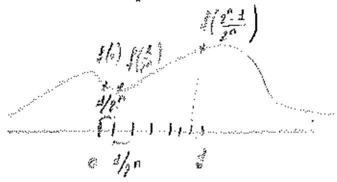
Reminders : * $Y_n \xrightarrow{d} Y$ if $F_{Y_n}(x_0) \to F_Y(x_0)$ as $n\to+\infty$, $\forall x_0$; $x \mapsto F_Y(x)$ is continuous at x_0.

$Y_n \xrightarrow{d} U(0,1)$ means that $Y_n \xrightarrow{d} Y$, where $Y \in U(0,1)$.

* continuity theorem : if $\varphi_{Y_n}(t) := \mathbb{E}[e^{itY_n}] \to \varphi_Y(t)$ as $n\to+\infty$, $\forall t\in\mathbb{R}$, where $t\mapsto \varphi_Y(t)$ is the characteristic function of some random variable Y, then $Y_n \xrightarrow{d} Y$.

$$\mathbb{E}[e^{itY_n}] = \sum_{j=0}^{2^n-1} e^{it\frac{j}{2^n}} \cdot \underbrace{\mathbb{P}(Y_n=\tfrac{j}{2^n})}_{=\frac{1}{2^n}} = \frac{1}{2^n}\sum_{j=0}^{2^n-1} e^{itj/2^n}$$. What is the limit of this sequence as $n\to+\infty$?

Let $f: [0,1]\to\mathbb{R}$ be a Riemann-integrable function.



$\rightarrow$ Riemann-integral : $\displaystyle\int_0^1 f(x)dx = \lim_n \sum_{j=0}^{2^n-1} f(\tfrac{j}{2^n})\cdot\frac{1}{2^n}$

Set $f(x) := e^{itx}$ $\forall x\in[0,1]$. Then $\displaystyle\int_0^1 e^{itx}dx = \int_0^1 f(x)dx = \lim_n \sum_{j=0}^{2^n-1} f(\tfrac{j}{2^n})\cdot\frac{1}{2^n} = \lim_n \sum_{j=0}^{2^n-1} e^{itj/2^n}\cdot\frac{1}{2^n} = \mathbb{E}[e^{itY_n}]$ $\forall n\geq 1$.

Also by definition, if $Y\in U(0,1)$, $f_Y(y) = \mathbb{1}_{[0,1]}(y)$ and $\mathbb{E}[e^{itY}] = \displaystyle\int e^{itx} f_Y(x)dx = \int_0^1 e^{itx}dx$

Hence $\lim_n \mathbb{E}[e^{itY_n}] = \mathbb{E}[e^{itY}]$, where $Y\in U(0,1)$ and by the continuity theorem,

$Y_n \xrightarrow{d} Y$.

QED

[T.K] 6.8.1.15 Let $(X_n)_{n\geq1}$ be a sequence of independent random variables such that $X_n \in La(a)$ $\forall n\geq 1$ and let $N\in Po(m)$, $m>0$ be a random variable independent of the sequence $(X_n)_{n\geq1}$.

Set $S_N := X_1 + X_2 + \dots + X_N$ and $S_0 = 0$.

Let $m\to+\infty$ and $a\to 0$ in such a way that $ma^2\to 1$.

To show : $S_N \xrightarrow{d} \mathcal{N}(0,2)$

Reminders : * $X\in La(a), a>0$ if $f_X(x) = \frac{1}{2a} e^{-|x|/a}$ $\forall x\in\mathbb{R}$.

* $X\in Po(m)$, $m>0$ if $p_X(k) = e^{-m}\frac{m^k}{k!}$, $\forall k\geq 0$.

We will proceed as before : $\varphi_{S_N}(t) = \mathbb{E}[e^{itS_N}] = \displaystyle\sum_{k\geq0}\mathbb{E}[e^{itS_k}/N=k]\,\mathbb{P}(N=k) \overset{ind}{=} \sum_{k\geq0}\mathbb{E}[e^{itS_k}]\,\mathbb{P}(N=k)$

$$= \sum_{k\geq0}\mathbb{E}\Big[\prod_{i=1}^{k} e^{itX_i}\Big]\,\mathbb{P}(N=k) \overset{X_k \text{ are iid}}{=} \sum_{k\geq0}\big(\mathbb{E}[e^{itX_1}]\big)^k\,\mathbb{P}(N=k) \qquad (*)$$

Now $\mathbb{E}[e^{itX_1}] = \displaystyle\int e^{itx}\frac{1}{2a}e^{-|x|/a}dx = \int_{-\infty}^{0} e^{itx}\frac{1}{2a}e^{x/a}dx + \int_0^{+\infty} e^{itx}\frac{1}{2a}e^{-x/a}dx$

$$= \frac{1}{2a}\left(\int_{-\infty}^{0} e^{x(it+\frac{1}{a})}\,dx + \int_{0}^{+\infty} e^{x(it-\frac{1}{a})}\,dx\right) = \frac{1}{2a}\cdot\underbrace{\frac{1}{it+\frac{1}{a}}\, e^{x(it+\frac{1}{a})}\Big|_{-\infty}^{0}}_{=\frac{1}{it+\frac{1}{a}}} + \frac{1}{2a}\underbrace{\frac{1}{it-\frac{1}{a}}\, e^{x(it-\frac{1}{a})}\Big|_{0}^{+\infty}}_{=-\frac{1}{it-\frac{1}{a}}}$$

$$= \frac{1}{2a}\cdot\frac{1}{it+\frac{1}{a}} - \frac{1}{2a}\cdot\frac{1}{it-\frac{1}{a}} = \frac{1}{2ait+2} - \frac{1}{2ait-2} = \frac{2ait-2-(2ait+2)}{(2ait+2)(2ait-2)} = \frac{-4}{-4a^2t^2-4} = \frac{1}{a^2t^2+1}$$

Hence $(*) = \sum_{k\geq 0}\left(\frac{1}{a^2t^2+1}\right)^k e^{-m}\frac{m^k}{k!} = e^{-m}\cdot\sum_{k\geq 0}\frac{1}{k!}\left(\frac{m}{a^2t^2+1}\right)^k$

Now $\lim_{\substack{m\to+\infty\\ a\to 0\\ ma^2\to 1}} e^{-m}\sum_{k\geq 0}\frac{1}{k!}\left(\frac{m}{a^2t^2+1}\right)^k \underset{e^x=\sum_{k\geq 0}\frac{x^k}{k!}\ \forall x\in\mathbb{R}.}{=} \lim_{\substack{m\to+\infty\\ a\to 0\\ ma^2\to 1}} e^{-m}\cdot e^{m/a^2t^2+1} = \lim_{\substack{m\to+\infty\\ a\to 0\\ ma^2\to 1}} e^{m\overbrace{\left(\frac{1}{a^2t^2+1}-1\right)}^{\frac{-a^2t^2}{a^2t^2+1}}} = e^{-t^2}$

Now recall that if $X\in N(\mu,\sigma^2)$, $\mathbb{E}[e^{itX}] = e^{it\mu}\cdot e^{-\frac{1}{2}\sigma^2t^2}$ $\Rightarrow X\in N(0,2)$ iff $\mathbb{E}[e^{itX}] = e^{-t^2}$

$\Rightarrow S_N \xrightarrow{d} N(0,2)$ by the continuity theorem.

QED.

[T.K]. 6.8.1.16 Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that $X_n\in Po(\mu)$ $\forall n\geq 1$.

Let $N\in Po(\lambda)$, $\lambda>0$ be independent of $(X_n)_{n\geq 1}$.

Set $S_N = X_1+X_2+\ldots+X_N$ and $S_0=0$.

Let $\lambda\to+\infty$ and $\mu\to 0$ so that $\lambda\mu\to\nu>0$.

To show: $S_N\xrightarrow{d} Po(\nu)$

As before, $\varphi_{S_N}(t) = \mathbb{E}[e^{itS_N}] \overset{ind}{=} \sum_{k\geq 0}\mathbb{E}[e^{itS_k}]\,\mathbb{P}(N=k) \overset{X_n\ iid}{=} \sum_{k\geq 0}\left(\mathbb{E}[e^{itX_1}]\right)^k\mathbb{P}(N=k)$ $(*)$

$\mathbb{E}[e^{itX_1}] = \sum_{k\geq 0} e^{itk}\cdot e^{-\mu}\cdot\frac{\mu^k}{k!} = e^{-\mu}\cdot\sum_{k\geq 0}\frac{1}{k!}(e^{it}\mu)^k \underset{e^x=\sum_{k\geq 0}\frac{x^k}{k!}}{=} e^{-\mu}\cdot e^{e^{it}\mu} = e^{\mu(e^{it}-1)}$

$\Rightarrow (*) = \sum_{k\geq 0} e^{k\mu(e^{it}-1)}\cdot e^{-\lambda}\cdot\frac{\lambda^k}{k!} = e^{-\lambda}\cdot\sum_{k\geq 0}\frac{1}{k!}\left(\lambda e^{\mu(e^{it}-1)}\right)^k = e^{-\lambda}\cdot e^{\lambda e^{\mu(e^{it}-1)}} = e^{\lambda(e^{\mu(e^{it}-1)}-1)}$

We have to determine the limit:

By continuity of $x\mapsto e^x$, it is enough to look at $\lim_{\substack{\lambda\to+\infty\\ \mu\to 0\\ \lambda\mu\to\nu}} \lambda\left(e^{\mu(e^{it}-1)}-1\right)$

$\lim_{\substack{\lambda\to+\infty\\ \mu\to 0\\ \lambda\mu\to\nu}} \lambda\left(e^{\mu(e^{it}-1)}-1\right) \underset{e^x=\sum_{k\geq 0}\frac{x^k}{k!}}{=} \lim_{\substack{\lambda\to+\infty\\ \mu\to 0\\ \lambda\mu\to\nu}} \lambda\left(\sum_{k\geq 0}\frac{(\mu(e^{it}-1))^k}{k!}-1\right) = \lim_{\substack{\lambda\to+\infty\\ \mu\to 0\\ \lambda\mu\to\nu}} \lambda\sum_{k\geq 1}\mu^k\frac{(e^{it}-1)^k}{k!}$

$$= \lim_{\substack{\lambda\to+\infty\\ \mu\to 0\\ \lambda\mu\to\nu}} \underbrace{\lambda\mu(e^{it}-1)}_{\downarrow\ \nu(e^{it}-1)} + \underbrace{\sum_{k\geq 1}\lambda\mu^k\frac{(e^{it}-1)^k}{k!}} = \nu(e^{it}-1)$$

$$\lambda\mu^k = \underset{\downarrow\,\nu}{(\lambda\mu)}\cdot\underset{\downarrow\,0}{\mu^{k-1}} \to 0$$

Hence $\lim_{\substack{\lambda\to+\infty\\ \mu\to 0\\ \lambda\mu\to\nu}} e^{\lambda(e^{\mu(e^{it}-1)}-1)} = e^{\nu(e^{it}-1)} = \mathbb{E}[e^{itY}]$, where $Y \in Po(\nu)$.

QED.

[T.K] 6.8.1.17. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that $X_n \in Po(\mu)$, $\mu>0$ $\forall n\geq 1$.
Let $N \in Ge(p)$ be independent of $(X_n)_{n\geq 1}$. Set $S_N = X_1+X_2+\dots+X_N$ and $S_0=0$.
Let $\mu\to 0$ and $p\to 0$ in a way such that $\frac{p}{\mu}\to\alpha>0$.

To show: $S_N \xrightarrow{d} Ge\left(\frac{\alpha}{\alpha+1}\right)$

As before. $\varphi_{S_N}(t) = \mathbb{E}[e^{iS_N t}] \overset{X_n \text{ iid}+N \text{ ind}}{=} \sum_{k\geq 0}(\mathbb{E}[e^{iX_1 t}])^k\,\mathbb{P}(N=k) = \sum_{k\geq 0}(e^{\mu(e^{it}-1)})^k(1-p)^k p$

$$= p\sum_{k\geq 0}\left(e^{\mu(e^{it}-1)}(1-p)\right)^k = \frac{p}{1-e^{\mu(e^{it}-1)}(1-p)} \quad (*)$$

We have to determine the limit.

$$\frac{1-e^{\mu(e^{it}-1)}(1-p)}{p} = \frac{1}{p}-\sum_{k\geq 0}\frac{(\mu(e^{it}-1))^k(1-p)}{p} = \frac{1}{p}-\left(\frac{1-p}{p}+\underbrace{\frac{\mu(e^{it}-1)(1-p)}{p}}_{\to\frac{1}{\alpha}(e^{it}-1)}+\underbrace{\sum_{k\geq 2}\frac{(\mu(e^{it}-1))^k(1-p)}{p}}_{\to 0 \text{ since } \mu^{k-1}\cdot\frac{\mu}{p}(e^{it}-1)\to 0}\right)$$

$$\to 1-\frac{1}{\alpha}(e^{it}-1) = \frac{\alpha-(e^{it}-1)}{\alpha}$$

Hence $(*) \to \frac{\alpha}{\alpha-(e^{it}-1)} = \frac{\alpha}{\alpha+1-e^{it}}$. But if $X \in Ge\left(\frac{\alpha}{\alpha+1}\right)$, $\varphi_X(t) = \sum_{k\geq 0}(e^{itk})\left(\frac{\alpha}{\alpha+1}\right)\left(1-\frac{\alpha}{\alpha+1}\right)^k$

$$= \left(\frac{\alpha}{\alpha+1}\right)\cdot\sum_{k\geq 0}\left(e^{it}\left(1-\frac{\alpha}{\alpha+1}\right)\right)^k = \frac{\alpha}{\alpha+1}\cdot\frac{1}{1-e^{it}\left(1-\frac{\alpha}{\alpha+1}\right)} = \frac{\alpha}{\alpha+1-e^{it}(\alpha+1-\alpha)} = \frac{\alpha}{\alpha+1-e^{it}}$$

Hence by the continuity theorem, $S_N \xrightarrow{d} Ge\left(\frac{\alpha}{\alpha+1}\right)$.

QED

[T.K] 6.8.2.7 Let $(X_n)_{n\geq 1}$ be a sequence of positive iid random variables with $\mathbb{E}[X_n]=1$ and $Var[X_n]=\sigma^2$.

Set $S_n := X_1 + \dots + X_n \quad \forall n \geq 1$.

To show $\sqrt{S_n} - \sqrt{n} \xrightarrow{d} \mathcal{N}(0, \frac{\sigma^2}{4})$, as $n \to +\infty$.

$X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} a \Rightarrow \frac{X_n}{Y_n} \to \frac{X}{a}$ $\quad X_n + Y_n \to X + a$

$X_n \cdot Y_n \to X \cdot a \quad X_n - Y_n \to X - a$

Idea: relate this to the central limit theorem and use slutsky's theorem to conclude the convergence in distribution.

By the central limit theorem, $\frac{S_n - n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$. Also, $S_n - n = (\sqrt{S_n} - \sqrt{n})(\sqrt{S_n} + \sqrt{n})$

$$\Rightarrow \sqrt{S_n} - \sqrt{n} = \frac{S_n - n}{\sqrt{S_n} + \sqrt{n}} = \frac{\frac{S_n - n}{\sqrt{n}}}{\frac{\sqrt{S_n} + \sqrt{n}}{\sqrt{n}}}$$

Now $\frac{S_n - n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$

$$\frac{\sqrt{S_n} + \sqrt{n}}{\sqrt{n}} = 1 + \sqrt{\frac{S_n}{n}} \xrightarrow{P} 2$$

Law of large numbers + $\sqrt{\cdot}$ is continuous

slutsky
$$\Rightarrow \sqrt{S_n} - \sqrt{n} \xrightarrow{d} \frac{1}{2}\mathcal{N}(0, \sigma^2) = \mathcal{N}(0, \frac{\sigma^2}{4})$$

QED

Remark on slutsky: $X_n \in U[-1,1] \; \forall n \geq 1$, $X \in U[-1,1]$ $\Rightarrow X_n \xrightarrow{d} X \in U[-1,1]$

$X \stackrel{d}{=} -X \in U[-1,1]$ and $X_n \xrightarrow{d} -X \in U[-1,1]$.

However $X_n + X_n \not\xrightarrow{d} X - X \stackrel{d}{=} 0$ as $n \to +\infty$. $\rightarrow$ other types of convergence have more information on the limit!

[T.K] 6.8.3.1. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables such that $X_n \in Be(p_n) \; \forall n \geq 1$.

Set $S_n = X_1 + X_2 + \dots + X_n$

To show: $\frac{1}{n}(S_n - \sum_{k=1}^{n} p_k) \xrightarrow{P} 0$ as $n \to +\infty$.

Fix $\varepsilon > 0$.

$Var(aX) = a^2 Var(X)$

X_k iid

$$\mathbb{P}\left(\left|\frac{1}{n}(S_n - \sum_{k=1}^{n} p_k)\right| > \varepsilon\right) \leq \frac{Var\left[\frac{1}{n}(S_n - \sum_{k=1}^{n} p_k)\right]}{\varepsilon^2} = \frac{1}{n^2\varepsilon^2}\sum_{k=1}^{n} Var(X_k - p_k) \quad (*)$$

$$\mathbb{E}\left[\frac{1}{n}(S_n - \sum_{k=1}^{n} p_k)\right] = \frac{1}{n}\sum_{k=1}^{n}(\mathbb{E}[X_k] - p_k) = 0.$$

$\mathbb{E}[X_k] = 1 \cdot \mathbb{P}(X_k = 1) + 0 \cdot \mathbb{P}(X_k = 0) = p_k$

what is the limiting behaviour of $(*)$? $Var[X_k - p_k] = \mathbb{E}[(X_k - p_k)^2] = (1 - p_k) \cdot p_k \leq 1 \quad \forall k \geq 1$.

$X_k - p_k$ centered

$$\Rightarrow (*) \leq \frac{1}{n^2\varepsilon^2}\sum_{k=1}^{n} 1 = \frac{n}{n^2\varepsilon^2} = \frac{1}{n\varepsilon^2} \to 0 \text{ as } n \to +\infty$$

We have shown convergence in probability.

QED